A LARGE-SAMPLE KOLMOGOROV-SMIRNOV TEST FOR NORMALITY OF EXPERIMENTAL ERROR IN A RANDOMIZED BLOCK DESIGN

by Constance L. Wood

BU-598-M December, 1976

SUMMARY

Considered is the limiting distribution of the Kolmogorov-Smirnov statistic, based on the estimated residuals in a Randomized Block Design, under the null hypothesis of normality of the experimental errors. Extended are the results of Kac, Kiefer, and Wolfowitz (1955) for the one-sample statistic and the results of Serfling and Wood (1974) for the Completely Randomized Design. Quantiles of Monte Carlo simulations of the limiting distributions for various numbers of treatments are presented.

Some Key Words: Kolmogorov-Smirnov statistics; Randomized Block Design; normality; limit distributions.

Paper No. BU-598-M in the Biometrics Unit Mimeograph Series, Department of Plant Breeding and Biometry, Cornell University, Ithaca, New York 14853.

This work was supported by Hatch Project 402.

A LARGE-SAMPLE KOLMOGOROV-SMIRNOV TEST FOR NORMALITY OF EXPERIMENTAL ERROR IN A RANDOMIZED BLOCK DESIGN

by Constance L. Wood

Biometrics Unit, Department of Plant Breeding and Biometry, Cornell University, Ithaca, New York 14853

SUMMARY

Considered is the limiting distribution of the Kolmogorov-Smirnov statistic, based on the estimated residuals in a Randomized Block Design, under the null hypothesis of normality of the experimental errors. Extended are the results of Kac, Kiefer, and Wolfowitz (1955) for the one-sample statistic and the results of Serfling and Wood (1974) for the Completely Randomized Design. Quantiles of Monte Carlo simulations of the limiting distributions for various numbers of treatments are presented.

Some Key Words: Kolmogorov-Smirnov statistics; Randomized Block Design; normality; limit distributions.

1. INTRODUCTION

The one-sample Kolmogorov-Smirnov test for normality with estimated mean and variance has received considerable attention. Early work dates back to the large-sample results of Kac, Kiefer, and Wolfowitz (1955). Lilliefors (1967) conducted a Monte Carlo study of the null hypothesis distribution of the Kolmogorov-Smirnov statistic for small samples. For a comprehensive review, see Durbin (1973).

In contrast residual analysis for data arising from experimental designs has been mainly limited to (i) examination of data plots; e.g., half-normal plots of estimated residuals and (ii) goodness-of-fit tests based on orthonormal transformations of the residuals. While data plots are appealing, the dependence among the plotted residuals invalidates the usual statistical interpretations. The second approach eliminates the dependence among the residuals, resulting in a reduced number of uncorrelated, homoscedastic variables to which many goodness-of-fit tests can be applied. Unfortunately, the transformations and, hence, the resulting variables are not unique. Of special interest here are the transformations given by Federer, Robson, and Tukey (1962) for the Randomized Block Design (R.B.D.) which are dependent on the orderings of treatments and blocks.

A third approach is that of Serfling and Wood (1974) who extended results of Kac, et al. (1955) to a test for normality of experimental error in a Completely Randomized Design. In particular it was shown that the Kolmogorov-Smirnov statistic, computed from the estimated within treatment residuals, asymptotically has the same null distribution as in the one-sample case. Here we consider a Kolmogorov-Smirnov statistic computed from the usual estimated residuals in a R.B.D. The advantages of this approach include (i) the test statistic is independent of the treatment and block orderings; (ii) the test statistic is easily computed; and (iii) the test is based on the same number of variables as original observations.

2. TEST PROCEDURE

Consider a R.B.D. with a treatments and b blocks. Suppose that the n = ab observations can be described by the following linear model:

$$Y_{ij} = \mu + \alpha_i + \beta_j + \epsilon_{ij}, \quad i = 1, \cdots, a; \; j = 1, \cdots, b, \qquad (2.1)$$

where α_i $(i = 1, \cdots, a)$ is the nonstochastic effect of the i^{th} treatment subject to the restriction $\sum_{i=1}^{a} \alpha_i = 0$; $\{\beta_j : j = 1, \cdots, b\}$ are independent and identically distributed (i.i.d.) block effects with $E(\beta_j) = 0$, $j = 1, \cdots, b$; and

$\{\epsilon_{ij}: i = 1, \cdots, a;\ j = 1, \cdots, b\}$ are i.i.d. experimental errors with common cumulative distribution function (c.d.f.) F. We want to consider the large sample $(b \to \infty)$ test of the hypothesis

$$H_0\colon F(x) = \Phi(x/\sigma), \text{ for some } \sigma > 0, \tag{2.2}$$

where Φ represents the standard normal c.d.f. In fact, (2.2) is equivalent to the hypothesis that $\{\epsilon_{ij}: i = 1, \cdots, a;\ j = 1, \cdots, b\}$ are i.i.d. $N(0, \sigma^2)$ for some σ^2.

The realized experimental errors are not observable and are usually estimated by $\{Y_{ij} - \bar{Y}_{i\cdot} - \bar{Y}_{\cdot j} + \bar{Y}_{\cdot\cdot}: i = 1, \cdots, a;\ j = 1, \cdots, b\}$ where $\bar{Y}_{i\cdot}$ is the i^{th} treatment mean, $\bar{Y}_{\cdot j}$ is the j^{th} block mean, and $\bar{Y}_{\cdot\cdot}$ is the overall mean. Our test procedure, is based on

$$\begin{aligned} e_{ij} &= \sqrt{\frac{a}{a-1}}\,(Y_{ij} - \bar{Y}_{\cdot j} - \bar{Y}_{i\cdot} + \bar{Y}_{\cdot\cdot}) \\ &= \sqrt{\frac{a}{a-1}}\,(\epsilon_{ij} - \bar{\epsilon}_{\cdot j} - \bar{\epsilon}_{i\cdot} + \bar{\epsilon}_{\cdot\cdot}), \quad i = 1, \cdots, a;\ j = 1, \cdots, b. \end{aligned} \tag{2.3}$$

The term $(a/a - 1)^{\frac{1}{2}}$ is a normalizing constant. In fact, if σ^2 were known, under H_0 each (e_{ij}/σ) would be normally distributed with zero mean and variance $(b - 1)/b$. This implies that asymptotically $\Phi(e_{ij}/\sigma)$ is uniform on $[0,1]$ if, and only if, (2.2) holds. Since σ^2 is unknown and not specified in (2.2), it will be estimated by

$$\hat{\sigma}^2 = \left[\sum_{i=1}^{a} \sum_{j=1}^{b} (Y_{ij} - \bar{Y}_{\cdot j} - \bar{Y}_{i\cdot} + \bar{Y}_{\cdot\cdot})^2\right] \Big/ (a - 1)(b - 1), \tag{2.4}$$

the Residual Mean Square.

The procedure is to first calculate the sample c.d.f. of $\{\Phi(e_{ij}/\hat{\sigma}): i = 1, \cdots, a;\ j = 1, \cdots, b\}$; i.e.,

$$G_n(t) = \left[\# \text{ of } \Phi[e_{ij}/\hat{\sigma}] \leq t\right]/n, \quad 0 \leq t \leq 1. \tag{2.5}$$

The Kolmogorov-Smirnov statistic is then given by

$$D_n = \sup_{0 \le t \le 1} \sqrt{n}\,|G_n(t) - t|. \tag{2.6}$$

Large values of D_n indicate disagreement between the Uniform c.d.f. and G_n, its unbiased estimator; and, hence, lead to rejection of (2.2). In fact the asymptotic rejection region is given by the quantiles of the limiting distribution of D_n. Letting $\Phi(x, y, \rho)$ denote $\Pr[X \le x, Y \le y]$, where (X, Y) have a Bivariate Normal Distribution $(\mu_X = 0,\ \mu_Y = 0,\ \sigma_X^2 = 1,\ \sigma_Y^2 = 1,\ \sigma_{XY} = \rho)$, we have

THEOREM: *Under* H_0, *the limiting distribution of* D_n *is that of* $\sup_{0 \le t \le 1} |V^\circ(t)|$, *where* V° *is the Gaussian Process with*

$$E[V^\circ(t)] = 0, \quad 0 \le t \le 1,$$

and, for $0 \le s,t \le 1$,

$$E[V^\circ(t)V^\circ(s)] = \min(s,t) - st + (a - 1)\left\{\Phi[\Phi^{-1}(s),\ \Phi^{-1}(t),\ -(a - 1)^{-1}] - st\right\}$$

$$- \tfrac{1}{2}\left(\frac{a}{a - 1}\right)\left\{\Phi^{-1}(t)\Phi^{-1}(s)\Phi'[\Phi^{-1}(t)]\Phi'[\Phi^{-1}(s)]\right\}. \tag{2.7}$$

Note that the covariance function of V° and, hence, the limiting distribution of D_n depend upon a, the number of treatments. Empirically generated, estimated quantiles for these distributions are given in Table 1.

[INSERT TABLE 1]

Asymptotic quantiles for the one-sample Kolmogorov-Smirnov statistic with estimated mean $(\bar{X})$ and estimated variance (S^2) are given under $a = \infty$. This follows from Theorem 1, Wood and Aref (1976) which states

$$\sup_{-\infty<x,y<\infty} \left|\rho^{-1}[\Phi(x, y, \rho) - \Phi(x)\Phi(y)] - \Phi'(x)\Phi'(y)\right| = 0,$$

and Theorem 1, Serfling and Wood (1974) which gives the limiting distribution of the one-sample Kolmogorov-Smirnov statistic as the supremum of the absolute value of a Gaussian process with zero mean function and covariance function similar to that given in (2.7) with $\{-\Phi'[\Phi^{-1}(t)]\Phi'[\Phi^{-1}(s)]\}$ replacing

$$(a - 1)^{-1}\left\{\Phi[\Phi^{-1}(s),\ \Phi^{-1}(t),\ -(a - 1)^{-1}] - st\right\} .$$

Monte Carlo methods were used to simulate the limiting distributions given in Table 1. The procedure was to approximate the Gaussian Process $V^\circ = V^\circ(a)$ by its finite-dimensional distributions, corresponding to evaluation of the process at 119 equally spaced points in the unit interval. A thousand multivariate normal random vectors with this covariance structure were generated using a program from the International Mathematical and Statistical Library. The empirical distributions for the supremum of the absolute value of the resulting multivariate normal vectors were then tabulated, thus approximating the limit law of D_n.

3. PROOF OF THE THEOREM

First we define the modified empirical stochastic process on $D[0,1]$ (see Billingsley (1968), Section 18) with the Skorohod topology; i.e.,

$$V_n(t) = \sqrt{n}\ [G_n(t) - t], \quad 0 \le t \le 1. \tag{3.1}$$

From (2.6), it follows that $D_n = \sup_{0\le t\le 1} |V_n(t)|$. Our approach is to show that V_n converges weakly to V° in $D[0,1]$. Since the sample paths of V° are continuous with probability one and the sup norm is a continuous functional on the space of all continuous functions, $C[0,1]$, the Theorem then follows from the continuous mapping theorem [Billingsley (1968), Section 5].

In order to show the weak convergence of V_n, we will first approximate it with

$$\Delta_n^*(t) = n^{-\frac{1}{2}} \sum_{i=1}^{a} \sum_{j=1}^{b} \{I[\Phi(Z_{ij}) \leq t] - t\}$$
$$+ n^{\frac{1}{2}} \Phi'[\Phi^{-1}(t)]\Phi^{-1}(t)[(\hat{\sigma}_n/\sigma) - 1], \quad 0 \leq t \leq 1, \tag{3.2}$$

where I(A) denotes the indicator of the event A and

$$Z_{ij} = [a/(a - 1)]^{\frac{1}{2}}(Y_{ij} - \bar{Y}_{\cdot j} - \alpha_i)/\sigma$$
$$= [a/(a - 1)]^{\frac{1}{2}}(\epsilon_{ij} - \bar{\epsilon}_{\cdot j})/\sigma, \quad i = 1, \cdots, a; \ j = 1, \cdots, b.$$

Then we will investigate the asymptotic properties of Δ_n^*, as $n \to \infty$. First we consider an approximation to $\hat{\sigma}_n$ and the asymptotic normality (AN) of $\hat{\sigma}_n$.

LEMMA 1. *Under* H_0,

$$\text{(i)} \quad \sqrt{ab}\,[(\hat{\sigma}_n/\sigma) - (2\sqrt{ab})^{-1} \sum_{i=1}^{a} \sum_{j=1}^{b} Z_{ij}^2] \xrightarrow{p} 0, \tag{3.3}$$

and

$$\text{(ii)} \quad \sqrt{b}\,[(\hat{\sigma}_n/\sigma) - 1] \ \textit{is} \ \ AN(0, [2(a - 1)]^{-1}). \tag{3.4}$$

PROOF. From (2.4), without loss of generality we may assume $\sigma^2 = 1$. First write

$$\sqrt{ab}\,(\hat{\sigma}_n - 1) = \sqrt{ab}\,(\hat{\sigma}_n^2 - 1)/(\hat{\sigma}_n + 1). \tag{3.5}$$

But $\hat{\sigma}_n^2$ is a consistent estimator of σ^2. Therefore, $\hat{\sigma}_n^2 \xrightarrow{p} 1$ as $n \to \infty$, implying that $(\hat{\sigma}_n + 1) \xrightarrow{p} 2$ as $n \to \infty$. From Slutsky's Theorem (Billingsley (1968), Theorem 4.1) it follows that $\sqrt{ab}\,(\hat{\sigma}_n - 1)$ has the same limiting law as $\sqrt{ab}\,(\hat{\sigma}_n^2 - 1)/2$. Now

$$\sqrt{ab}\ \hat{\sigma}_n^2 = \sqrt{ab}\left[\sum_{i=1}^{a}\sum_{j=1}^{b}(Y_{ij} - \bar{Y}_{\cdot j} - \bar{Y}_{i\cdot} + \bar{Y}_{\cdot\cdot})^2\right] \Big/ (a - 1)(b - 1)$$

$$= \sqrt{ab}\left[\sum_{i=1}^{a}\sum_{j=1}^{b}(\epsilon_{ij} - \bar{\epsilon}_{\cdot j} - \bar{\epsilon}_{i\cdot} + \bar{\epsilon}_{\cdot\cdot})^2\right] \Big/ (a - 1)(b - 1)$$

$$= \sqrt{ab}\left[\sum_{i=1}^{a}\sum_{j=1}^{b}(\epsilon_{ij} - \bar{\epsilon}_{\cdot j})^2\right] \Big/ (a - 1)(b - 1) \tag{3.6}$$

$$+ \sqrt{ab}\left[\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{\epsilon}_{i\cdot} - \bar{\epsilon}_{\cdot\cdot})^2\right] \Big/ (a - 1)(b - 1)$$

$$= [b/(b - 1)]^{\frac{1}{2}}(\sqrt{ab})^{-1} \sum_{j=1}^{b}\sum_{i=1}^{a} Z_{ij}^2$$

$$+ [\sqrt{ab}/(b - 1)]\left[\sum_{i=1}^{a} b(\bar{\epsilon}_{i\cdot} - \bar{\epsilon}_{\cdot\cdot})^2\right] \Big/ (a - 1).$$

But $[\sum_{i=1}^{a} b(\bar{\epsilon}_{i\cdot} - \bar{\epsilon}_{\cdot\cdot})^2]/(a - 1)$ is the Treatment Mean Square under the null hypothesis of "no treatment effects" and is distributed as $\chi^2_{(a-1)}/(a-1)$, for all $n = 1, 2, \cdots$. Since $\sqrt{b}/(b - 1) \to 0$ as $b \to \infty$, Slutsky's Theorem, (3.5), and (3.6) yield (i).

The most direct way to show (ii) is to note that $\sqrt{ab}\ (\hat{\sigma}_n^2 - 1)/2$ is distributed as

$$[ab/2(a - 1)(b - 1)]^{\frac{1}{2}}\left[\frac{\chi^2_{(a-1)(b-1)} - (a - 1)(b - 1)}{\sqrt{2(a - 1)(b - 1)}}\right].$$

But $[\chi^2_{(a-1)(b-1)} - (a - 1)(b - 1)]/[2(a - 1)(b - 1)]^{\frac{1}{2}}$ is AN(0,1) as $b \to \infty$. Therefore, $\sqrt{ab}\,(\hat{\sigma}^2_n - 1)$ is AN(0,1) and $\sqrt{b}\,(\hat{\sigma}^2_n - 1)/2$ is AN(0, $[2(a - 1)]^{-1}$). This and (3.5) imply (ii).

Now we are prepared to show the asymptotic equivalence in probability of V_n and Δ^*_n.

LEMMA 2. *Under* H_0, $V_n - \Delta^*_n \xrightarrow{p} 0$ *in* D[0,1].

PROOF. In analogy with (3.1) and (3.2), for each i ($i = 1, \cdots, a$) and $0 \le t \le 1$, define

$$V_{ni}(t) = b^{-\frac{1}{2}} \sum_{j=1}^{b} \left\{ I[\Phi(e_{ij}/\hat{\sigma}_n) \le t] - t \right\}$$

$$= b^{-\frac{1}{2}} \sum_{j=1}^{b} \Big\{ I\Big[\Phi(Z_{ij}) \le \Phi[\Phi^{-1}(t)(\hat{\sigma}_n - \sigma)/\sigma \qquad (3.7)$$

$$+ \big(a/(a - 1)\big)^{\frac{1}{2}}(\bar{Y}_{i.} - \bar{Y}_{..} - \alpha_i)/\sigma]\Big] - t\Big\},$$

and

$$\Delta^*_{ni}(t) = b^{-\frac{1}{2}} \sum_{j=1}^{b} \Big\{ I[\Phi(Z_{ij}) \le t] - t$$

$$+ \Phi'[\Phi^{-1}(t)][\Phi^{-1}(t)(\hat{\sigma}_n - \sigma)/\sigma \qquad (3.8)$$

$$+ \big(a/(a - 1)\big)^{\frac{1}{2}}(\bar{Y}_{i.} - \bar{Y}_{..} - \alpha_i)/\sigma]\Big\}.$$

We can write

$$V_{ni}(t) = \Delta_{ni}[\phi_{ni}(t)], \quad 0 \le t \le 1,$$

where

$$\Delta_{ni}(t) = b^{-\frac{1}{2}} \sum_{j=1}^{b} \left\{ I[\Phi(Z_{ij}) \le t] - t \right\}, \quad 0 \le t \le 1,$$

and

$$\phi_{ni}(t) = \Phi\left[\Phi^{-1}(t)(\hat{\sigma}_n - \sigma)/\sigma + \left(a/(a-1)\right)^{\frac{1}{2}}(\bar{Y}_{i.} - \bar{Y}_{..} - \alpha_i)/\sigma\right], \quad 0 \le t \le 1.$$

Note that ϕ_{ni} is a *random change of time* in the sense of Billingsley (1968), Section 17.

Now for fixed i ($i = 1, \cdots, a$), $\{Z_{ij}: j = 1, \cdots, b\}$ are i.i.d. $N(0,1)$. Also, under H_0, $\left(ab/(a-1)\right)^{\frac{1}{2}}(\bar{Y}_{i.} - \bar{Y}_{..} - \alpha_i)/\sigma \sim N(0,1)$. Therefore, it follows from Lemma 1 (ii) and Serfling and Wood (1974), Lemma 2, that

$$\phi_{ni} \xrightarrow{p} I \text{ in } D[0,1], \tag{3.9}$$

where I is the identity function $I(t) \equiv t$ on $[0,1]$, and

$$\Delta_{ni} - \Delta^*_{ni} \xrightarrow{p} 0 \text{ in } D[0,1]. \tag{3.10}$$

Noting that

$$\begin{aligned} \sup_{0\le t\le 1} |\Delta^*_{ni}(t) - V_{ni}(t)| &= \sup_{0\le t\le 1} |\Delta^*_{ni}(t) - \Delta_n\left(\phi_n(t)\right)| \\ &\le \sup_{0\le t\le 1} |\Delta^*_{ni}\left(\phi_n(t)\right) - \Delta^*_{ni}(t)| \qquad (3.11) \\ &\quad + \sup_{0\le t\le 1} |\Delta_{ni}(t) - \Delta^*_{ni}(t)|, \end{aligned}$$

the tightness of Δ^*_{ni} (see Serfling and Wood (1974), (3.25)), (3.9) and (3.10) imply

$$V_{ni} - \Delta^*_{ni} \xrightarrow{p} 0 \text{ in } D[0,1]. \tag{3.12}$$

Since the number of treatments is fixed,

$$V_n(t) = a^{-\frac{1}{2}} \sum_{i=1}^{a} V_{ni}(t), \quad 0 \le t \le 1,$$

and

$$\Delta_n^*(t) = a^{-\frac{1}{2}} \sum_{i=1}^{a} \Delta_{ni}^*(t), \quad 0 \le t \le 1,$$

the result follows from the triangle inequality.

The following lemma will be useful in deriving the limiting distribution of Δ_n^* and, hence, V_n.

LEMMA 3. *Under* H_0, *for every* $j \ge 1$,

$$\mathrm{Var}\left[a^{-\frac{1}{2}} \sum_{i=1}^{a} (Z_{ij}^2 - 1)/2\right] = [a/2(a - 1)], \tag{3.13}$$

$$\begin{aligned} &\mathrm{Cov}\left[a^{-\frac{1}{2}} \sum_{i=1}^{a} (Z_{ij}^2 - 1)/2,\ a^{-\frac{1}{2}} \sum_{i=1}^{a} I[Z_{ij} \le \Phi^{-1}(t)]\right] \\ &= -[a/2(a - 1)]\Phi^{-1}(t)\Phi'[\Phi^{-1}(t)], \quad 0 \le t \le 1, \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} &\mathrm{Cov}\left[a^{-\frac{1}{2}} \sum_{i=1}^{a} I[Z_{ij} \le \Phi^{-1}(t)],\ a^{-\frac{1}{2}} \sum_{i=1}^{a} I[Z_{ij} \le \Phi^{-1}(s)], \quad 0 \le s,t \le 1,\right. \\ &= \min(s,t) - st + (a - 1)\left\{\Phi[\Phi^{-1}(t),\ \Phi^{-1}(s),\ -(a - 1)^{-1}] - st\right\}. \end{aligned} \tag{3.15}$$

PROOF. In order to show (3.13), we have only to note that

$$[(a-1)/a] \sum_{i=1}^{a} Z_{ij}^2 = \sum_{i=1}^{a} (\epsilon_{ij} - \bar{\epsilon}_{.j})^2$$

which is distributed as $\chi^2_{(a-1)}$. Therefore

$$\mathrm{Var}\left(a^{-\frac{1}{2}} \sum_{i=1}^{a} (Z_{ij}^2 - 1)/2\right) = (4a)^{-1}[a/(a-1)]^2 \mathrm{Var}\left\{[a/(a-1)] \sum_{i=1}^{a} Z_{ij}^2\right\}$$

$$= [a/2(a-1)].$$

Continuing with (3.14)

$$\mathrm{Cov}\left\{a^{-\frac{1}{2}} \sum_{i=1}^{a} (Z_{ij}^2 - 1)/2,\ a^{-\frac{1}{2}} \sum_{i=1}^{a} I[Z_{ij} \le \Phi^{-1}(t)]\right\}$$

$$= E\left\{(2a)^{-1} \sum_{i=1}^{a} (Z_{ij}^2 - 1) I[Z_{ij} \le \Phi^{-1}(t)]\right\}$$

$$+ E\left\{(2a)^{-1} \sum_{\substack{i,k \\ i \ne k}}^{a} (Z_{kj}^2 - 1) I[Z_{ij} \le \Phi^{-1}(t)]\right\} \tag{3.16}$$

$$= E\left\{(Z_{1j}^2 - 1) I[Z_{2j} \le \Phi^{-1}(t)]/2\right\}$$

$$+ (a-1) E\left\{(Z_{1j}^2 - 1) I[Z_{2j} \le \Phi^{-1}(t)]/2\right\}.$$

But

$$E\left\{Z_{1j}^2 I[Z_{2j} \le \Phi^{-1}(t)]\right\} = E\left\{I[Z_{2j} \le \Phi^{-1}(t)] E(Z_{1j}^2 | Z_{2j})\right\}.$$

Since Z_{1j} and Z_{2j} have a standard bivariate normal distribution, with correlation $\rho = -(a - 1)^{-1}$,

$$E[Z_{1j}^2 | Z_{2j}] = \mathrm{Var}(Z_{1j} | Z_{2j}) + [E(Z_{1j} | Z_{2j})]^2$$

$$= [1 - (a - 1)^{-2}] + [-(a - 1)^{-1} Z_{2j}]^2$$

$$= [1 - (a - 1)^{-2}] + Z_{2j}^2/(a - 1)^2.$$

Therefore,

$$E\{Z_{1j}^2 I[Z_{2j} \leq \Phi^{-1}(t)]\} = t[1 - (a - 1)^{-2}]$$

$$+ (a - 1)^{-2} E\{Z_{2j}^2 I[Z_{2j} \leq \Phi^{-1}(t)]\} \tag{3.17}$$

$$= t + (a - 1)^{-2} E\{Z_{2j}^2 I[Z_{2j} \leq \Phi^{-1}(t)] - t\}.$$

Noting that

$$E\left[Z_{2j}^2 I[Z_{2j} \leq \Phi^{-1}(t)] - t\right] = \int_{-\infty}^{\Phi^{-1}(t)} [x^2 - 1] d\Phi(x)$$

$$= - \int_{-\infty}^{\Phi^{-1}(t)} d[x\Phi'(x)] \tag{3.18}$$

$$= -\Phi^{-1}(t)\Phi'[\Phi^{-1}(t)],$$

(3.14) follows from (3.16) and (3.17). Finally,

$$\mathrm{Cov}\left(a^{-\frac{1}{2}} \sum_{i=1}^{a} \{I[Z_{ij} \leq \Phi^{-1}(t)] - t\},\ a^{-\frac{1}{2}} \sum_{i=1}^{a} \{I[Z_{ij} \leq \Phi^{-1}(s)] - s\}\right)$$

$$= a^{-1} \sum_{i=1}^{a} \sum_{k=1}^{a} E\left(\{I[Z_{ij} \leq \Phi^{-1}(t)] - t\} \{I[Z_{kj} \leq \Phi^{-1}(s)] - s\}\right)$$

$$= \min(s,t) - st + (a - 1)\left\{\Phi[\Phi^{-1}(t), \Phi^{-1}(s), -(a - 1)^{-1}] - st\right\},$$

where $\Phi(x, y, \rho)$ is as defined in Section 2.

Now we can show that V_n converges weakly to the Gaussian Process given in (2.7).

PROOF OF THEOREM. From Lemma 2, it suffices to show that Δ_n^* converges to the appropriate gaussian process. First we show that for all $k \geq 1$, the finite dimensional distributions of $\Delta_n^* \left(\Delta_n^*(t_1), \cdots, \Delta_n^*(t_k), \quad 0 \leq t_1, \cdots, t_k \leq 1\right)$ converge in distribution to a multivariate normal vector with covariance structure given by (2.7). Without loss of generality, assume $k = 2$.

Choose constants c_1, c_2 and $0 \leq t_1, t_2 \leq 1$. Consider

$$c_1\Delta_n^*(t_1) + c_2\Delta_n^*(t_2)$$

$$= c_1 b^{-\frac{1}{2}} \sum_{j=1}^{b}\left(a^{-\frac{1}{2}} \sum_{i=1}^{a}\left\{I[Z_{ij} \leq \Phi^{-1}(t_1)] - t_1\right\}\right)$$

$$+ c_1(ab)^{\frac{1}{2}}\Phi^{-1}(t_1)\Phi'[\Phi^{-1}(t_1)](\sigma_n - \sigma)/\sigma$$

$$+ c_2 b^{-\frac{1}{2}} \sum_{j=1}^{b}\left(a^{-\frac{1}{2}} \sum_{i=1}^{a}\left\{I[Z_{ij} \leq \Phi^{-1}(t_2)] - t_2\right\}\right)$$

$$+ c_2(ab)^{\frac{1}{2}}\Phi^{-1}(t_2)\Phi'[\Phi^{-1}(t_2)](\hat{\sigma}_n - \sigma)/\sigma$$

$$= b^{-\frac{1}{2}} \sum_{j=1}^{b}\left(c_1 a^{-\frac{1}{2}} \sum_{i=1}^{a}\left\{I[Z_{ij} \leq \Phi^{-1}(t_1)] - t_1\right\} + c_2 a^{-\frac{1}{2}} \sum_{i=1}^{a}\left\{I[Z_{ij} \leq \Phi^{-1}(t_2)] - t_2\right\}\right)$$

$$+ \left\{c_1\Phi^{-1}(t_1)\Phi'[\Phi^{-1}(t_1)] + c_2\Phi^{-1}(t_2)\Phi'[\Phi^{-1}(t_2)]\right\}(ab)^{\frac{1}{2}}(\hat{\sigma}_n - \sigma)/\sigma.$$

From Lemma 1(i), $c_1\Delta_n^*(t_1) + c_2\Delta_n^*(t_2)$ has the same limiting distribution as

$$X_b = b^{-\frac{1}{2}} \sum_{j=1}^{b} \Big(c_1 a^{-\frac{1}{2}} \sum_{i=1}^{a} \{I[Z_{ij} \le \Phi^{-1}(t_1)] - t_1\} + c_2 a^{-\frac{1}{2}} \sum_{i=1}^{a} \{I[Z_{ij} \le \Phi^{-1}(t_2)] - t_2\} \Big)$$

$$+ \{c_1 \Phi^{-1}(t_1)\Phi'[\Phi^{-1}(t_1)] + c_2 \Phi^{-1}(t_2)\Phi'[\Phi^{-1}(t_2)]\}(2\sqrt{ab})^{-1} \Big[\sum_{j=1}^{b} \sum_{i=1}^{a} (Z_{ij}^2 - 1)\Big]$$

$$= b^{-\frac{1}{2}} \sum_{j=1}^{b} W_j(t_1, t_2, c_1, c_2),$$

where

$$W_j(t_1, t_2, c_1, c_2) = c_1 a^{-\frac{1}{2}} \sum_{i=1}^{a} \{I[Z_{ij} \le \Phi^{-1}(t_1)] - t_1\} + c_2 a^{-\frac{1}{2}} \sum_{i=1}^{a} \{I[Z_{ij} \le \Phi^{-1}(t_2) - t_2\}$$

$$+ \{c_1 \Phi^{-1}(t_1)\Phi'[\Phi^{-1}(t_1)] + c_2 \Phi^{-1}(t_2)\Phi'[\Phi^{-1}(t_2)]\} a^{-\frac{1}{2}} \sum_{j=1}^{a} (Z_{ij}^2 - 1)/2.$$

With this representation, the asymptotic normality of X_b follows from the Lindberg-Levy Central Limit Theorem, if the $\mathrm{Var}(X_b)$ exists. But

$$\mathrm{Var}(X_b) = \mathrm{Var}(W_1)$$

$$= c_1^2 \,\mathrm{Var}\Big(a^{-\frac{1}{2}} \sum_{i=1}^{a} \{I[Z_{i1} \le \Phi^{-1}(t_1)] - t_1\}\Big)$$

$$+ c_2^2 \,\mathrm{Var}\Big(a^{-\frac{1}{2}} \sum_{i=1}^{a} \{I[Z_{i1} \le \Phi^{-1}(t_2)] - t_2\}\Big) \tag{3.19}$$

$$+ 2c_1c_2 \,\mathrm{Cov}\Big(a^{-\frac{1}{2}} \sum_{i=1}^{a} \{I[Z_{i1} \le \Phi^{-1}(t_1)] - t_1\},\ a^{-\frac{1}{2}} \sum_{i=1}^{a} \{I[Z_{i1} \le \Phi^{-1}(t_2) - t_2\}\Big)$$

$$+ \left\{c_1\Phi^{-1}(t_1)\Phi'[\Phi^{-1}(t_2)] + c_2\Phi^{-1}(t_2)\Phi'[\Phi^{-1}(t_2)]\right\}^2 \operatorname{Var}\left[a^{-\frac{1}{2}} \sum_{j=1}^{a} (Z_{i1}^2 - 1)/2\right]$$

$$+ 2c_1\left\{c_1\Phi^{-1}(t_1)\Phi'[\Phi^{-1}(t_1)] + c_2\Phi^{-1}(t_2)\Phi'[\Phi^{-1}(t_2)]\right\}$$

$$\cdot \operatorname{Cov}\left(a^{-\frac{1}{2}} \sum_{i=1}^{a} \left\{I[Z_{i1} \le \Phi^{-1}(t_1)] - t_1\right\},\ a^{-\frac{1}{2}} \sum_{i=1}^{a} (Z_{i1}^2 - 1)/2\right)$$

$$+ 2c_2\left\{c_1\Phi^{-1}(t_1)\Phi'[\Phi^{-1}(t_1)] + c_2\Phi^{-1}(t_2)\Phi'[\Phi^{-1}(t_2)]\right\}$$

$$\cdot \operatorname{Cov}\left(a^{-\frac{1}{2}} \sum_{i=1}^{a} \left\{I[Z_{i1} \le \Phi^{-1}(t_2)] - t_2\right\},\ a^{-\frac{1}{2}} \sum_{i=1}^{a} (Z_{i1}^2 - 1)/2\right).$$

From Lemma 3,

$$\begin{aligned}
\operatorname{Var}(X_b) = {} & c_1^2\left(t_1(1 - t_1) + (a - 1)\left\{\Phi[\Phi^{-1}(t_1),\ \Phi^{-1}(t_1),\ -(a - 1)^{-1}] - t_1^2\right\}\right) \\
& + c_2^2\left(t_2(1 - t_2) + (a - 1)\left\{\Phi[\Phi^{-1}(t_2),\ \Phi^{-1}(t_2),\ -(a - 1)^{-1}] - t_2^2\right\}\right) \\
& + 2c_1c_2\left(\min(t_1,\ t_2) - t_1t_2 + (a - 1)\left\{\Phi[\Phi^{-1}(t_1),\ \Phi^{-1}(t_2),\ -(a - 1)^{-1}] - t_1t_2\right\}\right) \\
& + \left\{c_1\Phi^{-1}(t_1)\Phi'[\Phi^{-1}(t_1)] + c_2\Phi^{-1}(t_2)\Phi'[\Phi^{-1}(t_2)]\right\}^2[a/2(a - 1)] \\
& + 2c_1\left\{c_1\Phi^{-1}(t_1)\Phi'[\Phi^{-1}(t_1)] + c_2\Phi^{-1}(t_2)\Phi'[\Phi^{-1}(t_2)]\right\} \qquad (3.20) \\
& \quad \cdot \left\{-[a/2(a - 1)]\Phi^{-1}(t_1)\Phi'[\Phi^{-1}(t_1)]\right\} \\
& + 2c_2\left\{c_1\Phi^{-1}(t_1)\Phi'[\Phi^{-1}(t_1)] + c_2\Phi^{-1}(t_2)\Phi'[\Phi^{-1}(t_2)]\right\} \\
& \quad \cdot \left\{-[a/2(a - 1)]\Phi^{-1}(t_2)\Phi'[\Phi^{-1}(t_2)]\right\}
\end{aligned}$$

$$= c_1^2\Big(t_1(1 - t_1) + (a - 1)\{\Phi[\Phi^{-1}(t_1),\ \Phi^{-1}(t_1),\ -(a - 1)^{-1}] - t_1^2\}$$

$$- [a/2(a - 1)]\{\Phi^{-1}(t_1)\Phi'[\Phi^{-1}(t_1)]\}^2\Big)$$

$$+ c_2^2\Big(t_2(1 - t_2) + (a - 1)\{\Phi[\Phi^{-1}(t_2),\ \Phi^{-1}(t_2),\ -(a - 1)^{-1}] - t_2^2\}$$

$$- [a/2(a - 1)]\{\Phi^{-1}(t_2)\Phi'[\Phi^{-1}(t_2)]\}^2\Big)$$

$$+ 2c_1c_2\Big(\min(t_1,\ t_2) - t_1t_2 + (a - 1)\{\Phi[\Phi^{-1}(t_1),\ \Phi^{-1}(t_2),\ -(a - 1)^{-1}] - t_1t_2\}$$

$$- [a/2(a - 1)]\{\Phi^{-1}(t_1)\Phi^{-1}(t_2)\Phi'[\Phi^{-1}(t_1)]\Phi'[\Phi^{-1}(t_2)]\}\Big).$$

Also noting that $E[X_b] = E[W_1] = 0$, we have that X_b is $AN\big(0,\ Var(X_b)\big)$. Therefore, $c_1\Delta_n^*(t_1) + c_2\Delta_n^*(t_2)$ is $AN\big(0,\ Var(X_b)\big)$. From (3.20) and the Cramér-Wold Theorem, it follows that $\big(\Delta_n^*(t_1),\ \Delta_n^*(t_2)\big)'$ is asymptotically multivariate normal with covariance structure given by (2.7).

It remains to show that given ϵ and η, there exists $\eta > 0$ and N_0 such that

$$P[\sup_{|t-s|<\delta} |\Delta_n^*(t) - \Delta_n^*(s)| > \epsilon] < \delta, \quad n \geq N_0\ . \tag{3.21}$$

Noting that

$$P[\sup_{|t-s|<\delta} |\Delta_n^*(t) - \Delta_n^*(s)| > \epsilon] \leq P[\sup_{|t-s|<\delta} a^{-\frac{1}{2}} \sum_{i=1}^{a} |\Delta_{ni}^*(t) - \Delta_{ni}^*(s)| > \epsilon]$$

$$\leq \sum_{i=1}^{a} P[\sup_{|t-s|<\delta} |\Delta_{ni}^*(t) - \Delta_{ni}^*(s)| > a^{\frac{1}{2}}\epsilon],$$

(3.21) then follows from the tightness of Δ_{ni}^*, $i = 1, \cdots, a$, as given in Serfling and Wood (1974), (3.25). The weak convergence of the finite dimensional distributions of Δ_n^* and (3.21), imply the weak convergence of Δ_n^*, and hence, V_n to V°. (See Billingsley (1968), Theorem 8.1.)

4. REMARKS

It should be noted that this test procedure is also valid if Block X Treatment interaction exists but the effects are assumed to be normally and independently distributed with zero mean and variance component $\sigma^2_{\alpha\beta} > 0$. See Scheffé (1959). In this case, the interaction effects are absorbed into the experimental error. However, this test should have power against non-normal alternatives caused by the failure of the interaction effects to satisfy these assumptions; e.g., see Federer et al. (1962).

Secondly, in this paper, we have restricted attention to the Kolmogorov-Smirnov statistic. The weak convergence of V_n to V° can also be used to find the limiting distributions of other EDF statistics. See Stephens (1974).

Finally, the question of power and rate of convergence of the proposed test are not discussed here but will be deferred to another paper.

REFERENCES

Billingsley, P. (1968). *Convergence of Probability Measures*. Wiley, New York.

Durbin, J. (1973). Weak convergence of the sample distribution function when parameters are estimated. *Ann. Statist.* 1, 279-290.

Federer, W. T., Robson, D. S., and Tukey, J. W. (1962). Tests for non-additivity. Biometrics Unit Mimeo Series No. BU-147-M, Cornell University.

Kac, M., Kiefer, J., and Wolfowitz, J. (1955). On tests of normality and other tests of goodness of fit based on distance methods. *Ann. Math. Statist.* 26, 189-211.

Lilliefors, H. (1967). On the Kolmogorov-Smirnov test for normality with mean and variance unknown. *J. Am. Statist. Assoc.* 62, 399-402.

Scheffé, H. (1959). *The Analysis of Variance.* Wiley, New York.

Serfling, R. J. and Wood, C. L. (1974). On null-hypothesis limiting distributions of Kolmogorov-Smirnov type statistics with estimated location and scale parameters. Florida State University Statistics Report M339.

Stephens, M. A. (1974). EDF statistics for goodness-of-fit and some comparisons. *J. Am. Statist. Assoc.* 69, 730-737.